

Complex Geometry Exercises

Week 10

Exercise 1. Let (X, J) be a complex manifold. Show that $H^0(X, \tau_X)$ is naturally a complex Lie algebra, for which the Lie bracket is complex bilinear.

(**Hint:** Recall the Nijenhuis tensor $N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$.)

Exercise 2. Prove the following assertions.

- (i) The group $\text{Aut}(\mathbb{CP}^n)$ is path-connected.
- (ii) Any $f \in \text{Aut}(\mathbb{CP}^n)$, with $f \neq 1$, has $n + 1$ fixed points (counted with multiplicity).
- (iii) The group $\text{Aut}(\mathbb{CP}^1)$ is sharply 3-transitive.
- (iv) The group $\text{Aut}(\mathbb{CP}^n)$ is 2-transitive but not 3-transitive for $n \geq 2$.

Exercise 3.

- (i) Consider (E_1, h_1) and (E_2, h_2) holomorphic hermitian vector bundles over a Kähler manifold. Prove that

$$\mathcal{H}^{p,q}(E_1 \oplus E_2) \cong \mathcal{H}^{p,q}(E_1) \oplus \mathcal{H}^{p,q}(E_2) .$$

- (ii) Using the Euler sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{CP}^n} \rightarrow \bigoplus_{i=0}^n \mathcal{O}(1) \rightarrow \tau_{\mathbb{CP}^n} \rightarrow 0 ,$$

show that

$$\dim H^i(\mathbb{CP}^n, \tau_{\mathbb{CP}^n}) = \begin{cases} (n+1)^2 - 1 & \text{if } i = 0 , \\ 0 & \text{otherwise} \end{cases} .$$

(continues on the back)

Exercise 4. Let $f : X \rightarrow X$ be a continuous map. We say z is fixed by f if $f(z) = z$. For an isolated fix point, we define the index of z as the degree of the homology map induced in $U \setminus \{z\}$ for U a small enough neighbourhood of z .

Assume that f is holomorphic.

- (i) Show that the degree is always positive.
- (ii) Using local coordinates, show that the degree is equal to the multiplicity of the zero of the (locally defined) function $g(z) = f(z) - z$.

Assume that X is closed.

- (iii) Prove that if $f \neq \text{Id}$, the number of fixed points (with multiplicity) coincides with the Lefschetz number of f :

$$\Lambda_f = \sum_{k \geq 0} \text{Tr} \left(f^* : H^k(X, \mathbb{R}) \rightarrow H^k(X, \mathbb{R}) \right).$$

- (iv) Compute the number of fixed points if f is homotopic to the identity.
- (v) Study the fixed points of $f \in \text{Aut}(\mathbb{T}^n)$.